INTEGRATION OF POLYNOMIAL FUNCTIONS — TRANSCRIPT

Firstly, we will consider integration as the reverse process of differentiation.

Consider

$$y = x^3 + 2x^2 + 3x + 4$$

$$\Rightarrow \frac{dy}{dx} = 3x^2 + 4x + 3$$

So if we integrate $3x^2+4x+3$ with respect to x

written as $\int (3x^2+4x+3)\,dx$

the answer should be $x^3 + 2x^2 + 3x + 4$

However, the constant term on end, in this case $+4$, cannot be determined from $\frac{dy}{dx}$ and so in the integral it must appear as $+c$, an unknown constant.

Points to note:

The integral of 3 gives $3x$

More generally, the integral of a constant k gives kx

For powers of x, we reverse the rule for differentiation to give $\int ax^n\,dx = \frac{ax^{n+1}}{n+1} + c$

INTEGRATION OF POLYNOMIAL FUNCTIONS — TRANSCRIPT

Examples using $\int ax^n\,dx = \dfrac{ax^{n+1}}{n+1} + c$

$$\int x^2\,dx = \frac{x^3}{3} + c$$

$$\int x^7\,dx = \frac{x^8}{8} + c$$

$$\int 5x^4\,dx = \frac{5x^5}{5} + c = x^5 + c$$

$$\int -3x^5\,dx = -\frac{3x^6}{6} + c = -\frac{x^6}{2} + c$$

$$\int 7\,dx = 7x + c$$

$$\int 0\,dx = +c$$

$$\int (5x^2 - 6x + 4)\,dx = \frac{5x^3}{3} - \frac{6x^2}{2} + 4x + c = \frac{5}{3}x^3 - 3x^2 + 4x + c$$

$$\int (4x^5 - x^2 + 1)\,dx = \frac{4x^6}{6} - \frac{x^3}{3} + x + c = \frac{2x^6}{3} - \frac{x^3}{3} + x + c$$

$$\int (x^4 - x^3 + x^2 - x + 1)\,dx = \frac{x^5}{5} - \frac{x^4}{4} + \frac{x^3}{3} - \frac{x^2}{2} + x + c$$

$$\int \left(\frac{2}{3}x^2 + \frac{1}{4}x - \frac{1}{2}\right)dx = \frac{2x^3}{9} + \frac{x^2}{8} - \frac{x}{2} + c$$

Exam type question:

The gradient function of a curve is given by $\frac{dy}{dx} = 3x^2 + 4x - 5$. The curve passes through the point $(3, 4)$. Find the equation of the curve.

Solution:

$$\frac{dy}{dx} = 3x^2 + 4x - 5$$

$$\Rightarrow \quad y = \frac{3x^3}{3} + \frac{4x^2}{2} - 5x + c$$

$$y = x^3 + 2x^2 - 5x + c$$

Passes through $(3,4)$

so

$$4 = 3^3 + 2(3)^2 - 5(3) + c$$

$$4 = 27 + 18 - 15 + c$$

$$4 = 30 + c$$

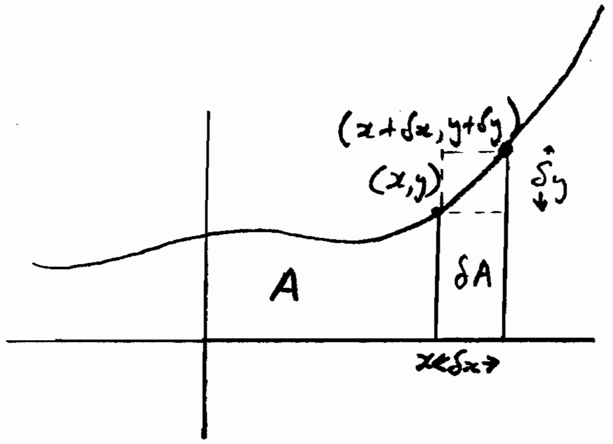
$$4 - 30 = c$$

$$-26 = c$$

$$\therefore \quad y = x^3 + 2x^2 - 5x - 26$$

Definite Integration

Area under a curve



A is the area under the curve measured between some arbitrary point on x-axis and the point with x-coordinate x

If we allow x to change by a small amount δx then y will change by a small amount δy, and A will change by a small amount δA.

δA is larger than the rectangle $y \times \delta x$ and smaller than the rectangle $(y+\delta y)\times \delta x$

$$\therefore \qquad y\,\delta x \leq \delta A \leq (y+\delta y)\,\delta x$$

Dividing by δx

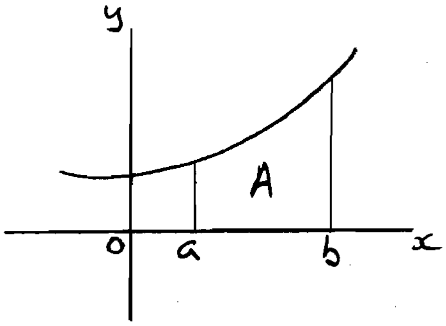
$$y \leq \frac{\delta A}{\delta x} \leq y + \delta y$$

Letting $\delta x \to 0$, $\Rightarrow \delta y \to 0$, $\dfrac{\delta A}{\delta x} \to \dfrac{dA}{dx}$

$$y \leq \frac{dA}{dx} \leq y$$

$$\Rightarrow \qquad \frac{dA}{dx} = y$$

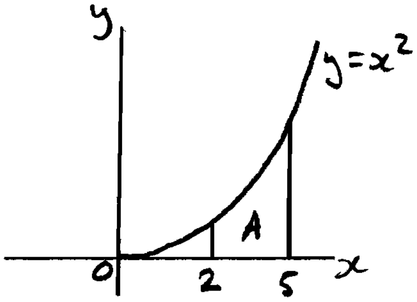
$$\therefore \quad A = \int y \, dx$$



For area A between $x = a$ and $x = b$ we write $A = \int_a^b y \, dx$

This is found by evaluating the integral at both a and b then subtracting to find the difference between the two. As the constant c is in both integrals it is normally omitted.

Example 1 Find the area between the curve $y = x^2$, the x-axis and the lines $x = 2$ and $x = 5$.



$$A = \int_2^5 y \, dx = \int_2^5 x^2 \, dx$$

$$= \left[\frac{x^3}{3} \right]_2^5$$

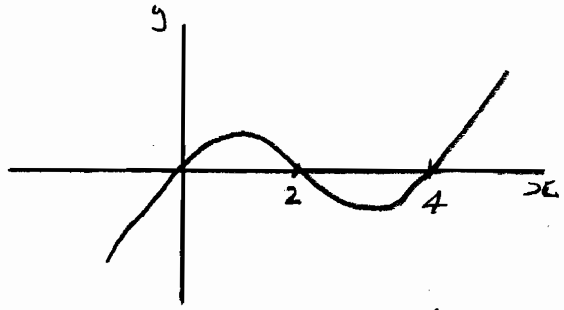
$$= \left(\frac{5^3}{3} \right) - \left(\frac{2^3}{3} \right)$$

$$= \frac{125}{3} - \frac{8}{3} = \frac{117}{3}$$

$$= 39 \text{ units}^2$$

Example 2

$$y = x(x-2)(x-4)$$
$$y = x(x^2 - 6x + 8)$$
$$y = x^3 - 6x^2 + 8x$$



Find area between curve and x axis between $x = 2$ and $x = 4$

$$A = \int_2^4 y\,dx = \int_2^4 (x^3 - 6x^2 + 8x)\,dx$$

$$= \left[\frac{x^4}{4} - \frac{6x^3}{3} + \frac{8x^2}{2} \right]_2^4$$

$$= \left[\frac{x^4}{4} - 2x^3 + 4x^2 \right]_2^4$$

$$= \left(\frac{4^4}{4} - 2(4)^3 + 4(4)^2 \right) - \left(\frac{2^4}{4} - 2(2)^3 + 4(2)^2 \right)$$

$$= (64 - 128 + 64) - (4 - 16 + 16)$$

$$= 0 - 4$$

$$= -4$$

$-$ sign indicates area is below x-axis

Area $= 4$ units^2