FACTOR AND REMAINDER THEOREMS

TRANSCRIPT

Consider $53 \div 6$

$= 8$ remainder 5

Note that

$$53 = 6 \times 8 + 5$$

Similarly,

Consider $77 \div 5$

$= 15$ remainder 2

and note that

$$77 = 5 \times 15 + 2$$

In algebraic long division we can use a similar result to establish the remainder theorem.

Remainder Theorem

When a polynomial $f(x)$ is divided by $(x-a)$ then the remainder is given by $f(a)$

Proof

When $f(x)$ is divided by $(x-a)$ there will in general be a remainder r say

We can write

$$f(x) = (x-a)g(x) + r$$

for some function $g(x)$

Now consider $f(a)$

$$f(a) = (a-a)g(a) + r$$
$$= 0 \times g(a) + r$$
$$= r$$

$\therefore$ $f(a)$ = the remainder when $f(x)$ is divided by $(x-a)$

Factor Theorem

If $(x-a)$ is a factor of $f(x)$ then $f(a)=0$

and $x=a$ is a root of the equation $f(x)=0$

Conversely, if $f(a)=0$ then $(x-a)$ is a factor of $f(x)$

Proof

This is just a special case of the remainder theorem.

If $(x-a)$ is a factor of $f(x)$ then the remainder $=0$

when $f(x)$ is divided by $(x-a)$

However, by the remainder theorem

$$f(a) = \text{the remainder}$$

so $$f(a) = 0$$

Conversely, if $f(a)=0$ then the remainder $=0$

when $f(x) = (x-a)g(x) + \text{remainder}$

which means $(x-a)$ is actually a factor of $f(x)$

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TRANSCRIPT

Using the remainder theorem

Ex1) Find the remainder when $f(x) = x^3 - x^2 + 5x + 3$ is divided by $(x-2)$

$$\text{remainder} = f(2) = 2^3 - 2^2 + 5(2) + 3$$

$$= 8 - 4 + 10 + 3$$

$$= 17$$

It is far more work to establish this result by algebraic long division

$$
\begin{array}{r|llll}
 & x^2 & + x & + 7 & \\
\hline
x-2 & x^3 - x^2 & + 5x & + 3 & \\
 & \underline{x^3 - 2x^2} & & & \\
 & \quad x^2 & + 5x & & \\
 & \quad \underline{x^2} & \underline{- 2x} & & \\
 & & 7x & + 3 & \\
 & & \underline{7x} & \underline{- 14} & \\
 & & & + 17 &
\end{array}
$$

$$\text{so } f(x) = (x-2)(x^2 + x + 7) + 17$$

Ex2) When $f(x) = x^3 - 2x^2 + kx - 3$ is divided by $(x-4)$ the remainder is 49. Find k

By remainder theorem $f(4) = \text{remainder}$

$$\therefore f(4) = 49$$

$$\Rightarrow \quad 4^3 - 2(4)^2 + 4k - 3 = 49$$

$$64 - 32 + 4k - 3 = 49$$

$$4k = 49 - 64 + 32 + 3$$

Ex2 cont)

$4k = 20$

$k = 5$

Using the factor theorem

Ex3) Let $f(x) = x^3 - 5x^2 + 2x + 8$

Use the factor theorem to fully factorise $f(x)$ and hence sketch the graph of $y = f(x)$

We only review numbers which are factors of $+8$

Potentially, we could consider $\pm 1, \pm 2, \pm 4, \pm 8$, but the three numbers must multiply to give $+8$

$f(1) = 1^3 - 5(1)^2 + 2(1) + 8$
$= 1 - 5 + 2 + 8 = 6$ ✗

$f(-1) = (-1)^3 - 5(-1)^2 + 2(-1) + 8$

$= -1 - 5 - 2 + 8 = 0$ ✓

$f(2) = 2^3 - 5(2)^2 + 2(2) + 8$

$= 8 - 20 + 4 + 8 = 0$ ✓

Now $f(-1) = 0 \Rightarrow (x+1)$ is a factor

$f(2) = 0 \Rightarrow (x-2)$ is a factor

$\therefore f(x) = (x+1)(x-2)(x \quad)$

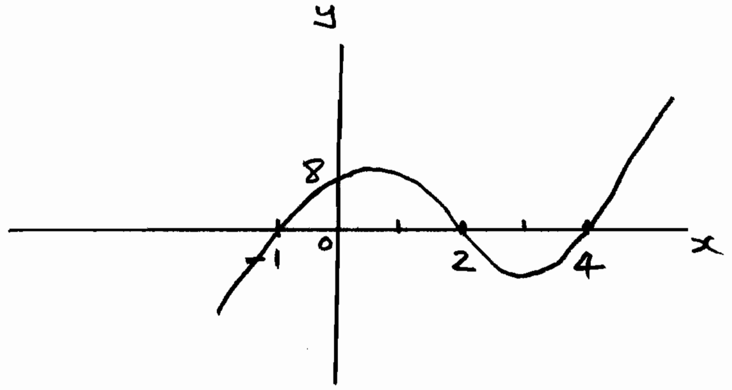
Since the three numbers in the brackets need to multiply to give $+8$, the missing number is -4

Ex3 cont) which means $f(4)$ should be 0

$$f(4) = 4^3 - 5(4)^2 + 2(4) + 8$$

$$= 64 - 80 + 8 + 8 = 0 \checkmark$$

$$\therefore \quad f(x) = (x+1)(x-2)(x-4)$$



Ex4) Use the factor theorem to establish a root of the equation $x^3 - 7x^2 + 13x - 3 = 0$

Use algebraic long division to help find the other two roots

Try numbers that are factors of ± 3

Let $f(x) = x^3 - 7x^2 + 13x - 3$

then $f(1) = 1^3 - 7(1)^2 + 13(1) - 3$

$$= 1 - 7 + 13 - 3 = 4 \quad \times$$

$$f(-1) = (-1)^3 - 7(-1)^2 + 13(-1) - 3$$

$$= -1 - 7 - 13 - 3 = -24 \quad \times$$

$$f(3) = 3^3 - 7(3)^2 + 13(3) - 3$$

$$= 27 - 63 + 39 - 3 = 0 \checkmark$$

$$\therefore (x-3) \text{ is a factor}$$

Ex4) cont) Perform long division

$$\begin{array}{r|l} & x^2 - 4x + 1 \\ \hline x-3 & x^3 - 7x^2 + 13x - 3 \\ & \underline{x^3 - 3x^2} \\ & \quad -4x^2 + 13x \\ & \quad \underline{-4x^2 + 12x} \\ & \qquad\quad + x - 3 \\ & \qquad\quad \underline{+ x - 3} \end{array}$$

$\therefore \quad f(x) = (x-3)(x^2-4x+1)$

If $f(x) = 0$

Either $x - 3 = 0 \quad \Rightarrow x = 3$

or $\quad x^2 - 4x + 1 = 0$

$$x = \frac{4 \pm \sqrt{16-4}}{2}$$

$$x = \frac{4 \pm \sqrt{12}}{2}$$

$$x = \frac{4 \pm 2\sqrt{3}}{2}$$

$$x = 2 \pm \sqrt{3}$$

Solution to $f(x) = 0$

$x = 3, \quad x = 2 + \sqrt{3}, \quad x = 2 - \sqrt{3}$

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