BINOMIAL EXPANSIONS — TRANSCRIPT

Pascal's Triangle

$$
\begin{array}{c}
1 \\
1 \quad 2 \quad 1 \\
1 \quad 3 \quad 3 \quad 1 \\
1 \quad 4 \quad 6 \quad 4 \quad 1 \\
1 \quad 5 \quad 10 \quad 10 \quad 5 \quad 1 \\
1 \quad 6 \quad 15 \quad 20 \quad 15 \quad 6 \quad 1 \\
1 \quad 7 \quad 21 \quad 35 \quad 35 \quad 21 \quad 7 \quad 1
\end{array}
$$

Ex 1

$$(a+b)^5$$

$$= a^5 + 5a^4b + 10a^3b^2 + 10a^2b^3 + 5ab^4 + b^5$$

Note that in every term the sum of the powers of a and b equals 5

Ex 2

$$(a+b)^6$$

$$= a^6 + 6a^5b + 15a^4b^2 + 20a^3b^3 + 15a^2b^4 + 6ab^5 + b^6$$

Pascal's triangle provides the binomial coefficient for each term. In this case 1, 6, 15, 20, 15, 6, 1

Ex 3

$$(2+3x)^3$$

$$= 2^3 + 3(2)^2(3x) + 3(2)(3x)^2 + (3x)^3$$

$$= 8 + 36x + 54x^2 + 27x^3$$

Ex 4

$$(2-x)^5$$

$$= (2)^5 + 5(2)^4(-x) + 10(2)^3(-x)^2 + 10(2)^2(-x)^3 + 5(2)(-x)^4 + (-x)^5$$

$$= 32 - 80x + 80x^2 - 40x^3 + 10x^4 - x^5$$

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Notice how the signs alternate in this example according to whether we have an odd or even powers of the negative term in the bracket.

NOTATION FOR COMBINATIONS

The number of ways of selecting r objects from n objects when order does not matter is given by:

$$\underbrace{\binom{n}{r} = {}^{n}C_{r} = {}_{n}C_{r}}_{\text{All these mean the same}} = \frac{n!}{r!(n-r)!}$$

where $n!$ (n factorial) $= n \times (n-1) \times (n-2) \times \ldots \times 2 \times 1$

eg $6! = 6 \times 5 \times 4 \times 3 \times 2 \times 1$

We also define $0! = 1$ to avoid inconsistency in our notation

Examples

1) $\binom{5}{2} = \frac{5!}{2!\,3!} = \frac{5 \times \cancel{4}^{2} \times \cancel{3} \times \cancel{2} \times 1}{\cancel{2} \times 1 \times \cancel{3} \times \cancel{2} \times 1} = 10$

2) $\binom{7}{5} = \frac{7!}{2!\,5!} = \frac{7.\cancel{6}^{3}}{\cancel{2}_{1}.1} = 21$

Here we are replacing multiplication signs with dots and using the largest factorial in the denominator to cancel out most of the numerator without even writing it down.

3) $\binom{6}{6} = \frac{6!}{6!\,0!} = \frac{6!}{6! \times 1} = 1$

This is an example using our definition of $0! = 1$

It makes sense because the number of ways of selecting 6 objects from 6 objects is obviously 1

We can now use combination notation to calculate the coefficients of terms within binomial expansions.

Ex 1 Find the coefficient of x^2 in $(1+2x)^{10}$

Term in x^2 will be $\binom{10}{2}(2x)^2(1)^8$

$$= \frac{\overset{5}{\cancel{10}}.9}{\underset{1}{\cancel{2}}.1} \times 4x^2 \times 1$$

$$= 45 \times 4x^2$$

$$= 180x^2$$

So the coefficient of x^2 is 180

<u>Why is $\binom{10}{2}$ involved in this calculation?</u>

This is because each term in the expansion has factors from each of the 10 brackets. The term in x^2 has 2 lots of $(2x)$ and 8 lots of (1) from the 10 brackets. However, there are $\binom{10}{2}$ ways of choosing the 2 lots of $(2x)$ from the

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10 brackets.

Ex 2

Find the coefficient of x^3 in $(2-x)^8$

Term will be $\binom{8}{3}(-x)^3(2)^5$

$$= \frac{8.7.\cancel{6}}{\cancel{3}.\cancel{2}.1}\left(-x^3\right)\times 32$$

$$= 56\times 32\times\left(-x^3\right)$$

$$= -1792x^3$$

$$\begin{array}{r} 56 \\ 32\times \\ \hline 112 \\ 1680 \\ \hline 1792 \\ \hline \end{array}$$

Coefficient of x^3 is -1792

Ex 3

Find coefficient of x^2 in $(1+2x)^{20}$

Term will be $\binom{20}{2}(2x)^2(1)^{18}$

$$= \frac{\overset{10}{\cancel{20}}.19}{\cancel{2}.1}\times 4x^2\times 1$$

$$= 190\times 4x^2$$

$$= 760x^2$$

Coefficient of x^2 is 760
